Imperial College London

Lecture 2

Signals in Time Domain

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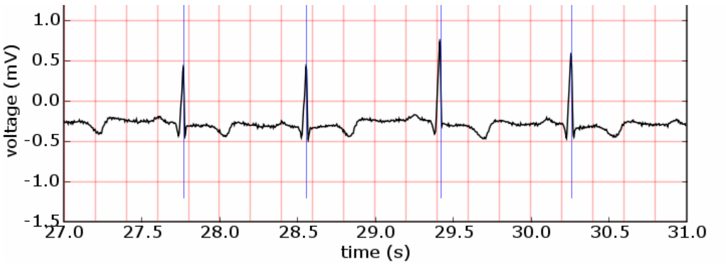
The first lecture is an introduction to signals from the time domain perspective. This lecture will be slightly longer than 50 minutes. The main focus is a revision of some of the materials covered last year, but I am taking a more mathematical modeling approach to signals with voltages expressed as a function of time.

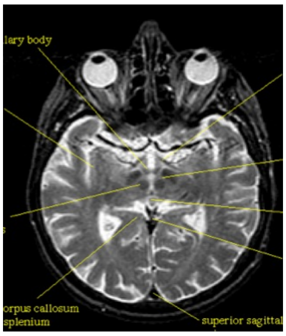
In the next lecture, I will take an alternative view, where signals will be considered not as functions of time, but of frequency.

Examples of signals

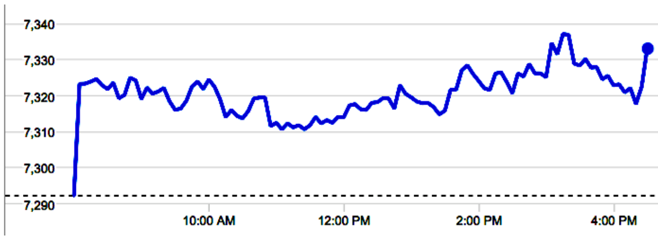
- Electrocardiogram (ECG) signal



- Magnetic Resonance Image (MRI) data as 2-dimensional signal



- FTSE 100 index in a day as signal (time series)



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Here are three examples of signals that we often encounter, and require some form of "processing". Firstly is the cardiac signal that your doctor may acquire. This is a **continuous time signal**, which is almost (but not exactly) periodic. The importance of this signal lies in the detail features appearing in the voltage vs time curve.

Another type of signal is actually NOT a real signal. For example, the plot of FTSE 100 index as it varies throughout the day is essentially numbers that are man-made, and it is **discrete** in nature, expressed as a sequence known as a time series. However, we often treat such a time series as a signal and apply the conventional processing techniques to perform prediction, analysis and the like!

Finally, shown here is a 2-dimensional MRI scan image of a brain. This is actually a function of intensity (of the image as pixels) in 2-D space . Therefore the independent variables are the x and y coordinate, and NOT time. However, signal processing techniques are applicable to such signals, not only as a function of distance (space), but also in 2 or more dimensions.

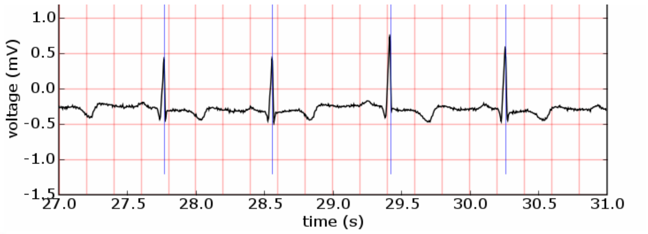
Signals Classification (1)

- Signals may be classified into:
  1. Continuous-time and discrete-time signals
  2. Analogue and digital signals
  3. Periodic and aperiodic signals
  4. Energy and power signals
  5. Deterministic and probabilistic signals
  6. Causal and non-causal
  7. Even and Odd signals

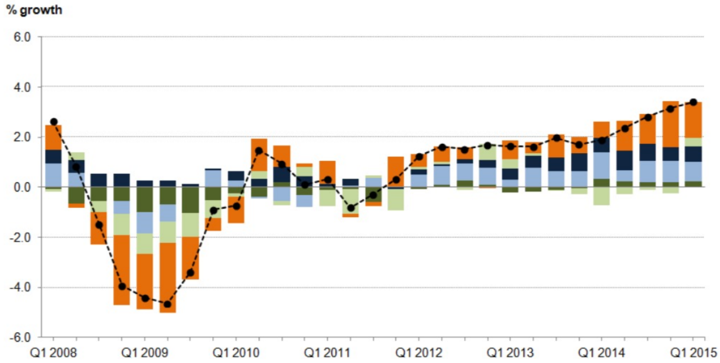
Here is 7 separate classifications of signals. Often such classification does not appear that useful. However, they are actually very important in signal processing because each class of signal has its own unique set of properties, significance and implications.

Signal Classification (2) – Continuous vs Discrete

- Continuous-time (CT), e.g. ECG signal



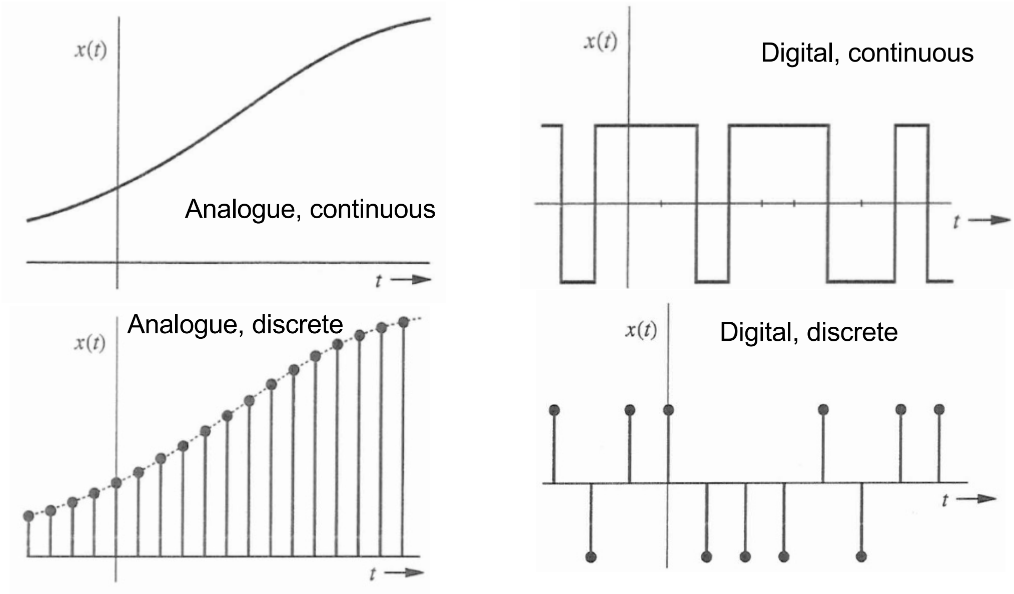
- Discrete-time (DT), e.g. UK growth rate



We have already looked at continuous time signal such as the ECG signal, and discrete time signal such as the stock market or the UK growth rate in the last few years.

Although real physical signals (such as ECG) are generally continuous in nature, we almost always process such as signal using computers. Therefore, in practice, signal processing are usually perform in the **discrete time domain**. The process of turning a continuous time signal to a discrete time signal is known as **sampling**. We will consider the mathematics relating to sampling in a later lecture.

Signal Classification (3) – Analogue vs Digital



Signals can be **analogue** or **digital**. Again most real signals are analogue in nature, but digital computers need to process this as numbers with discrete levels. The process of turning an analogue signal to a digital signal is through A-to-D converters.

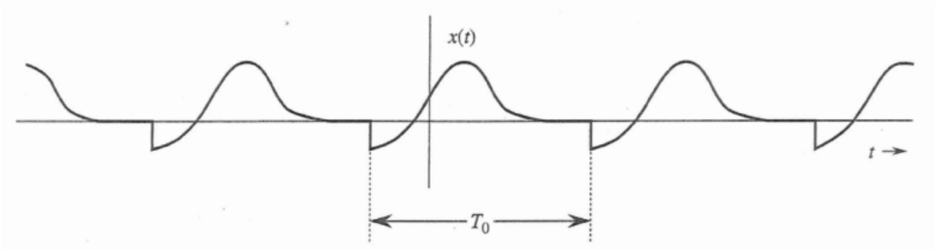
It is important to note that digitising an analogue signal introduces **error** (or distortion) and therefore it inherently a "corrupting" process. Digitizing a signal introduce **quantization noise**. In contrast, the process of sampling, done properly, will not corrupt the signal. We can always recover the original continuous time signal from the discrete time version perfectly. (At least this is theoretically possible).

Signal Classification (4) – Periodic vs Aperiodic

- A signal *x(t)* is said to be periodic if for some positive constant T_o

$$x(t) = x(t + T_0) \qquad \text{for all } t$$

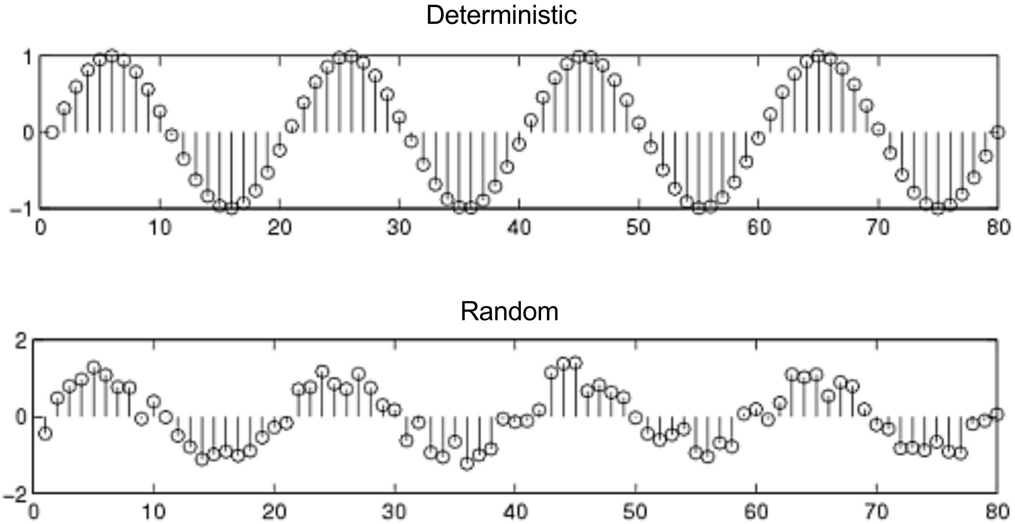
- The smallest value of T_o that satisfies the periodicity condition of this equation is the *fundamental period* of x(t).



Signals can be **periodic** or not. ECG is approximately periodic, and speech signal is definitely NOT periodic.

If a signal is periodic with period T_o, then it has a fundamental frequency $1/T_o$. An example of this is the note from a tuning fork – which is almost a perfect sinewave of a known frequency.

Signal Classification (5) – Deterministic vs Random



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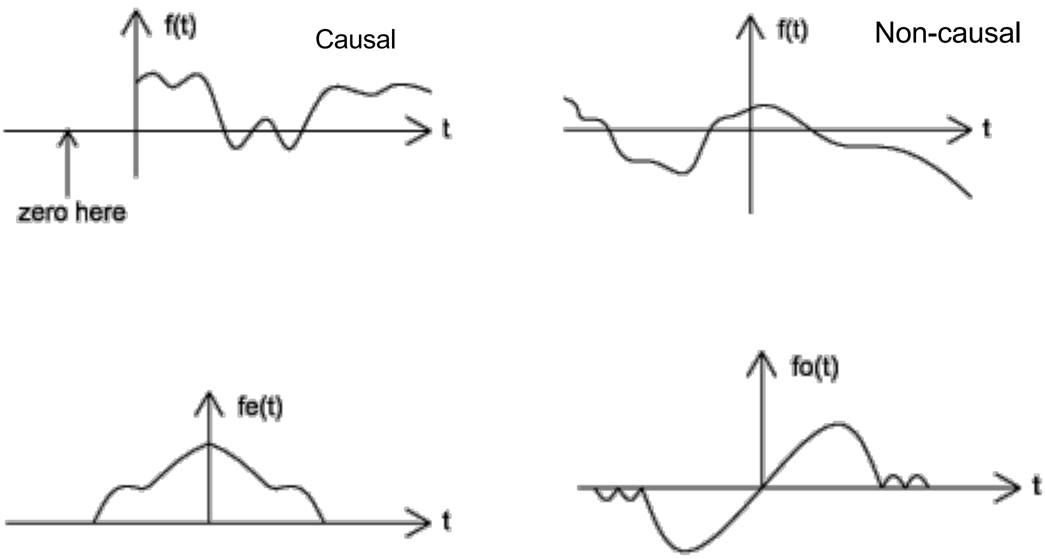
A signal can be **deterministic** or **random**.

Real signals are generally not completely deterministic, but many signals can be approximated by the sum of a deterministic component with random noise added. Often, the deterministic part of the signal is what you want to retain, and the random part is what you want to get rid of.

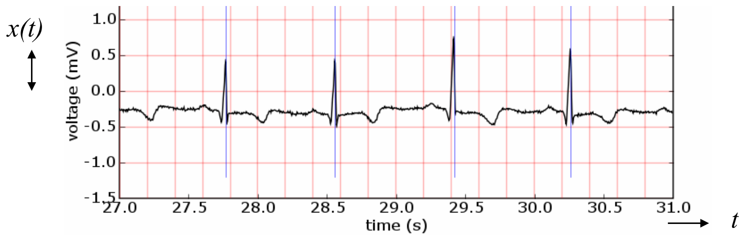
Signal Classification (6) – Causal/Non-causal, Even/Odd



Causal and **non-causal** simply refers to whether the signal has zero amplitude at time ≤ 0. If a signal x(t) = 0 for all t ≤ 0, it is known as causal. Otherwise, it is non-causal.

All real physical signals has a definite start and therefore it is causal. However, with the help of digital circuits and delay components, we actually can now processing signals and “pretend” that they are non-causal. We will see more of this later on in the course.

Size of a Signal *x(t)* as energy



- Measured by signal energy E_x:

$$E_x = \int_{-\infty}^{\infty} x^2(t)\ dt$$

- Generalize for a complex valued signal to:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\ dt$$

- Energy must be finite, which means

$$\text{signal amplitude } \to 0 \quad \text{as } |t| \to \infty$$

The first issue to consider when encountering a signal is to ask “how big is it?”

What is meant by “size” of a signal?

One useful measure of a signal size is its energy measure as defined here in the slide.

The square term (of voltage, say) ensures that the sign of the signal x(t) does not matter. (Otherwise, there is a danger that positive and negative parts of the signal cancel out each other.) The integration is over the duration of $\pm\infty$.

To be more general, the signal x(t) could be complex (i.e. with real and imaginary parts). What does a complex value mean? It means that the signal not only have magnitude, but also has phase information. For example if you are dealing with a sinusoidal signal, then the magnitude determines the signal amplitude (or peak value), and the phase determines the starting position at time 0.

Since the definition of energy of a signal requires integral over infinite time, this measure is only useful if the **energy is finite**. That is, as |t| → infinite, the signal amplitude must → 0.

Size of a Signal *x(t)* as power

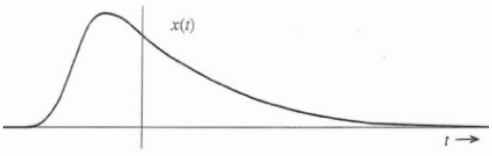
- If amplitude of *x(t)* does not → 0 when t → ∞, need to measure power *Px* instead:

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\, dt$$

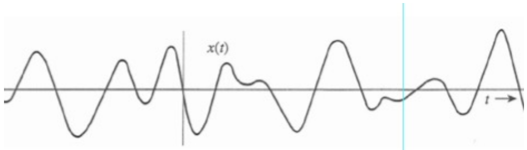
- Again, generalize for a complex valued signal to:

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt$$

- Signal with finite energy (zero power)



- Signal with finite power (infinite energy)



What happens if the signal does not have finite energy? What does this mean anyway?

For example, if you are considering the signal of the power mains from your household power socket. For all intend and purposes, the mains signal (50 Hz at 230V RMS) is continuous (i.e. goes on forever). Therefore when we consider the size of such as signal, we don't use energy – we use POWER instead as define above.

In other words,

POWER = ENERGY / TIME, and

ENERGY = POWER x TIME

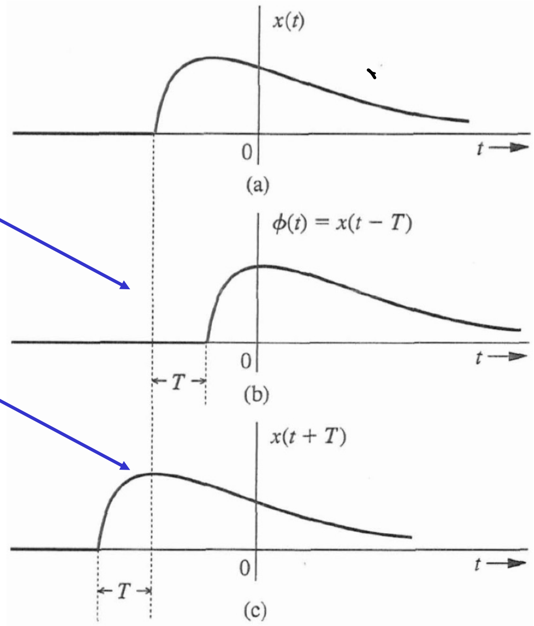
Useful Signal Operations –Time Shifting (1)

- Signal may be delayed by time T:

$$\phi\,(t + T) = x\,(t)$$

- or advanced by time T:

$$\phi\,(t - T) = x\,(t)$$



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When we consider signals as a function of time, there are a number of useful mathematical models that are being used very often.

Perhaps the most common is to express a signal with a certain time delay as shown above. Note that advancement in time is simply a delay of –T.

Useful Signal Operations –Time Scaling (2)

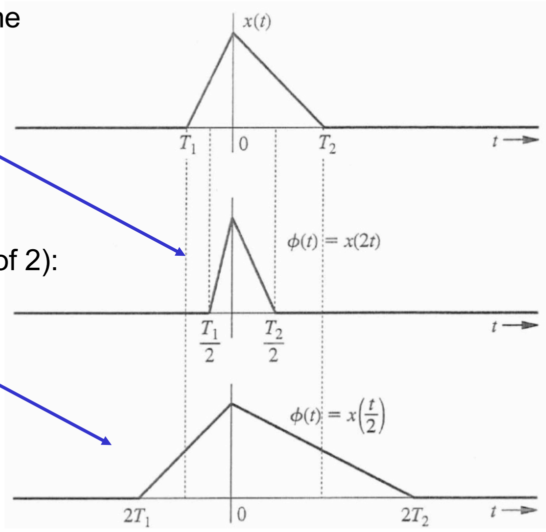
- Signal may be compressed in time (by a factor of 2):

$$\phi\,(t/2) = x(t)$$

- or expanded in time (by a factor of 2):

$$\phi\,(2t) = x(t)$$

- Same as recording played back at twice and half the speed respectively



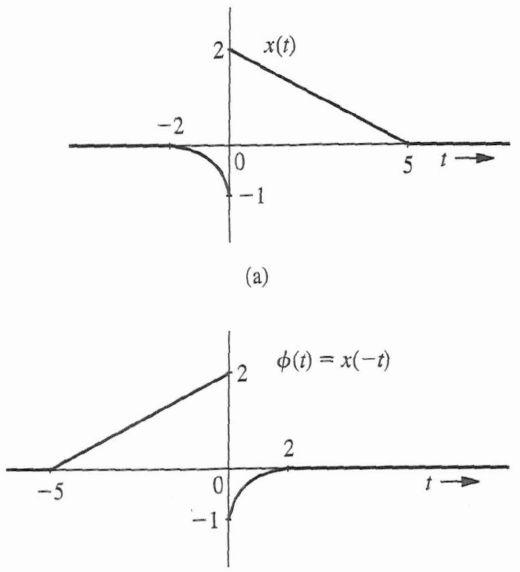
Another mathematical model we often use is the stretching and compression of a signal in time.

Useful Signal Operations –Time Reversal (3)

- Signal may be reflected about the vertical axis (i.e. time reversed):

$$\phi(t) = x(-t)$$

- We can combine these three operations.
- For example, the signal x(2t - 6) can be obtained in two ways:
  1. Delay x(t) by 6 to obtain x(t - 6), and then time-compress this signal by factor 2 (replace t with 2t) to obtain x (2t - 6).
  2. Alternately, time-compress x (t) by factor 2 to obtain x (2t), then delay this signal by 3 (replace t with t - 3) to obtain x (2t - 6).



The third common operation on a signal is time reversal. This may not appears that practical. (Who would play a tape back to front?)

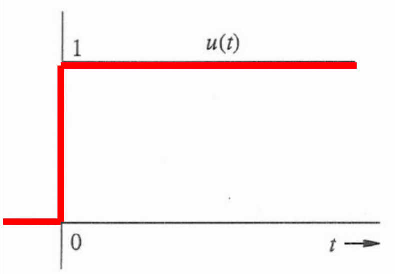
However, as you will find out later on the course when we consider a common signal processing operation known as "convolution", time-reversal plays a very important part.

Time reversal is achieved by simply reversing the sign of the time variable.

Signal Models (1) – Unit Step Function *u(t)*

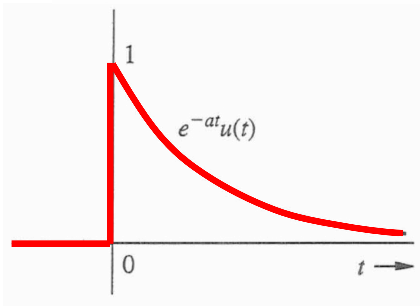
- Step function defined by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- Useful to describe a signal that begins at t = 0 (i.e. causal signal).
- For example, the signal e^{-at} represents an everlasting exponential that starts at t = -∞.
- The causal for of this exponential can be described as:

$$e^{-at}u(t)$$



Next let us consider a number of important time domain signals that will be use throughout this course.

Most important is the **step function** as shown here. Step signal is common – an instruction to a robot arm moving from A to B can be model as a step signal. As will be seen later on this course, the response of a system to a step signal input (known as the "**step response**") will characterise the entire system.

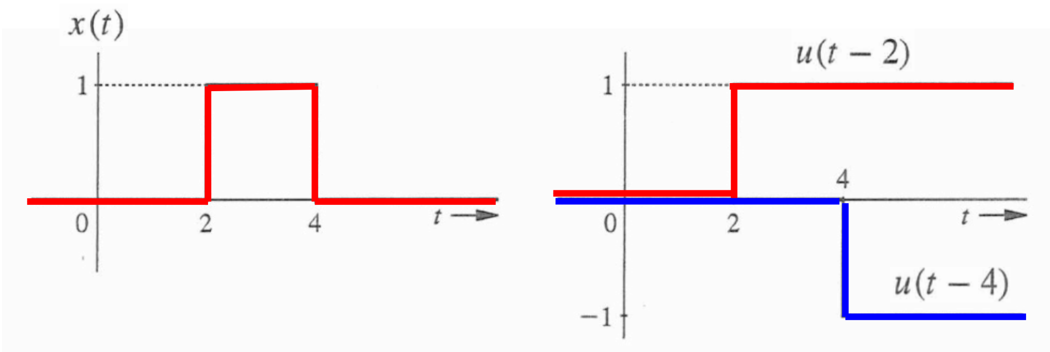
We often use the step function ***u(t)*** in modelling a causal signal. Here is a decay exponential that is causal. We simply multiply the exponential function with the step function!

$$e^{-at}u(t)$$

Signal Models (2) – Pulse signal

- A pulse signal can be presented by two step functions:

$$x(t) = u(t-2) - u(t-4)$$



Pulse signals are obvious. Less obvious is how to model this as the sum of two step functions with two different delays, one by 2 time units, and another by 4 time units:

$$x(t) = u(t-2) - u(t-4)$$

Signal Models (3) – Unit Impulse Function $\delta(t)$

- First defined by Dirac as:

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

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Impulse function is one of the most important functions in signal processing. It is sometimes known as the Dirac function, after the mathematician Paul Dirac.

It is also known as the **delta function** and is written as $\delta(t)$.

Unit impulse is a spike at t=0, and that its area is exactly = 1.

An impulse function can take on many other forms. For example, it can also be a pulse with with $\pm\varepsilon/2$, and the amplitude of the pulse is $1/\varepsilon$. It is centred at t = 0, and the area of the pulse (i.e. under the curve) is again exactly 1.

Multiplying a function $\Phi(t)$ by an Impulse

- Since impulse is non-zero only at t = 0, and $\Phi(t)$ at t = 0 is $\Phi(0)$, we get:

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

- We can generalise this for t = T:

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

If we have a time domain function $\phi(t)$ and multiply this with the impulse $\delta(t)$, we basically extract or sample the signal $\phi(t)$ at $t = 0$.

Therefore if we now delay the impulse function by T, then what we get is the value of $\phi(t)$ at $t = T$. In otherwise, we are sampling the function $\phi(t)$ at T. Therefore impulse function has a SAMPLING property.

Sampling Property of Unit Impulse Function

- Since we have: $\phi(t)\delta(t) = \phi(0)\delta(t)$
- It follows that: $$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)dt$$ $$= \phi(0)$$
- This is the same as "**sampling**" $\phi(t)$ at t = 0.
- If we want to sample $\phi(t)$ at t = T, we just multiple $\phi(t)$ with $\delta(t-T)$

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T)$$

- This is called the "**sampling property**" of the unit impulse.

Let us consider what happens when we multiply the unit impulse $\delta(t)$ by a function $\phi(t)$ that is continues at t = 0. Since the impulse has nonzero value only at t=0, and the value of $\phi(t)$ at t=0 is $\phi(0)$, we obtain:

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

In order words, multiplying a continuous function $\phi(t)$ with a unit impulse at t = 0 results in an impulse, also located at t=0 and has strength of $\phi(0)$.

We can now generalise this results by time-shifting the impulse function by delaying it by T. If you multiple $\phi(t)$ $\delta(t-T)$, which is an impulse located at t=T, we get:

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

Let us integrate this for t from $-\infty$ to $+\infty$, we get:

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T)$$

This result means that *the area under the product of a function with an impulse $\delta(t)$ is equal to the value of that function at the instant at which the unit impulse is located*. This property is known as the **sampling property of the unit impulse**.

The Exponential Function e^{st} (1)

- This exponential function is very important in signals & systems, and the parameter *s* is a complex variable given by:

$$s = \sigma + j\omega$$

- Therefore

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(cos\omega t + jsin\omega t) \quad \text{[eq 1]}$$

- Since $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^*t} = e^{(\sigma-j\omega)t} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(cos\omega t - jsin\omega t) \quad \text{[eq 2]}$$

- Eq 1 + Eq 2 gives:

$$e^{\sigma t}cos\omega t = \frac{1}{2}(e^{st} + e^{s^*t})$$

Another important function in the area of signals and systems is the exponential signal e^{st}, where s is complex in general, given by:

$$s = \sigma + j\omega$$

Substituting this provides the following important equation:

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(cos\omega t + jsin\omega t)$$

We can compare this exponential function e^{st} to the of the Euler's formula:

$$e^{j\omega t} = (cos\omega t + jsin\omega t)$$

Here the frequency variable $j\omega$ is genearlised to a complex variable $s = \sigma + j\omega$. For this reason, we designate the variable s as the ***complex frequency***.

The Exponential Function e^{st} (2)

- If $\sigma = 0$, then we have the function $e^{j\omega t}$, which has a real frequency of ω
- Therefore the complex variable $s = \sigma + j\omega$ is the **complex frequency**
- The function e^{st} can be used to describe a very large class of signals and functions. Here are a number of example:

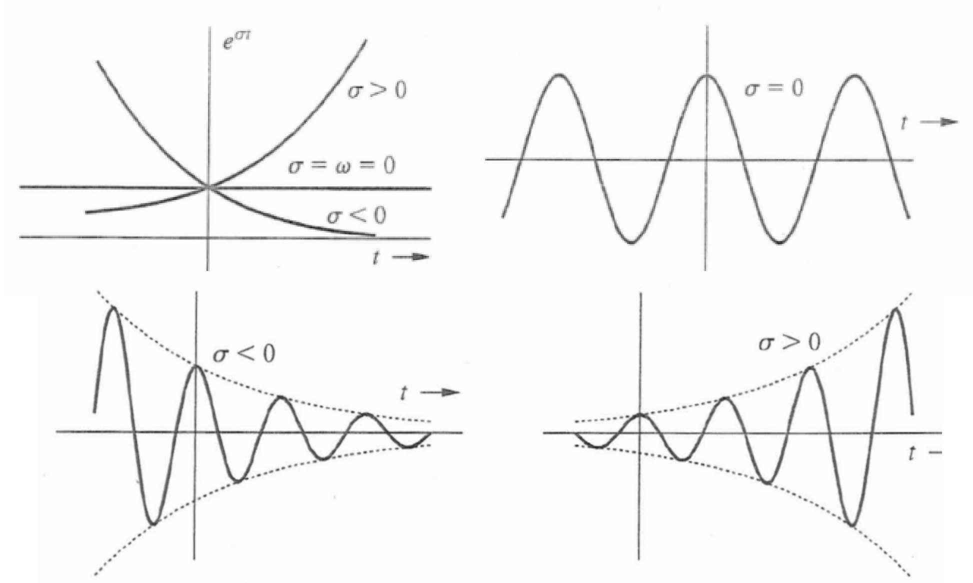
1. A constant $k = ke^{0t}$ $\quad (s = 0)$
2. A monotonic exponential $e^{\sigma t}$ $\quad (\omega = 0,\ s = \sigma)$
3. A sinusoid $\cos \omega t$ $\quad (\sigma = 0,\ s = \pm j\omega)$
4. An exponentially varying sinusoid $e^{\sigma t} \cos \omega t$ $\quad (s = \sigma \pm j\omega)$

This function is a very important. If $\sigma = 0$, then e^{st} is a sinusoidal function. It is used to represent steady state signal with a frequency ω.

If $\sigma \neq 0$, then the signal either grows or decay exponentially.

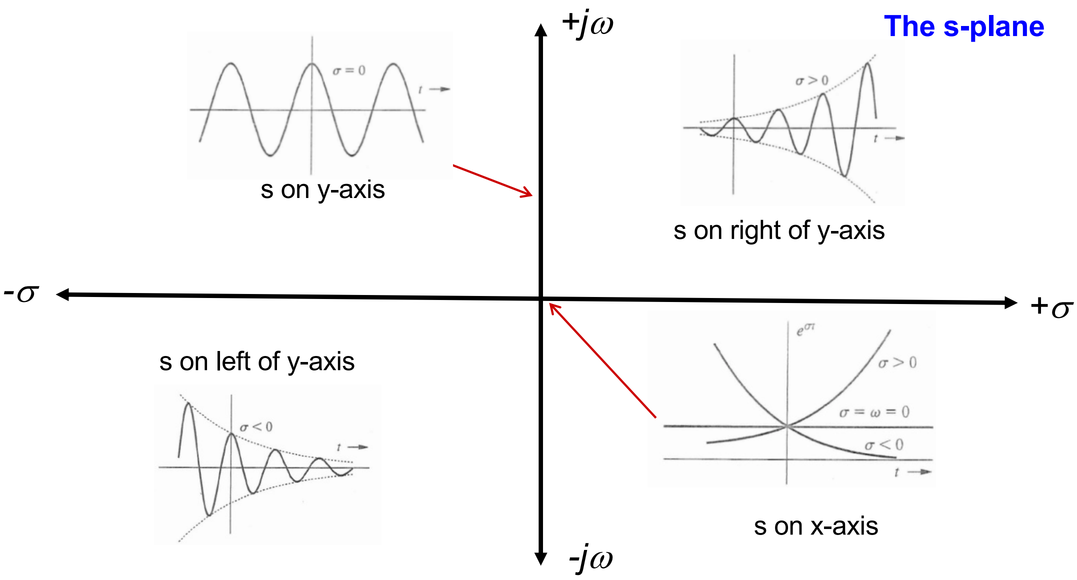
Laplace and Fourier transform, which we will study in later lectures, are based on this exponential function.

The Exponential Function e^{st} (2)



This four plots shows the four different possible signals represented by such an exponential function.

The Complex Frequency Plane $s = \sigma + j\omega$



Finally, one can express the value s (which is also known as "complex frequency", in a complex plane as shown here. We call this the s-plane. The location of the complex frequency of a signal will then take on the four different forms depending where s lies.

Three Big Ideas

1. The size of a time-limited signal is measured by it energy:

$$E_x = \int_{t_1}^{t_2} x^2(t)\, dt \qquad\qquad E_x = \sum_{n=1}^{N} x^2[n]$$

2. Delaying a signal x(t) by time T can be written as:

$$y(t) = x(t-T)$$

3. Unit impulse or delta function $\delta(t)$ can be use to model taking a sample from a signal. To take one sample of x(t) at time T is modelled as

$$x_T(t) = x(t) \times \delta(t-T)$$

Every session, I will try to identify three things that you MUST know if you forget everything else. I will call these the Big Ideas.

For today, these are:

1. Trying to determine the size of a signal is not as easy as you might think. You are probably familiar with using peak amplitude to measure the size. In the past, you have been exposed to the idea of "root-mean-square" or rms voltage. Here we define a term "energy" to measure the size of a signal. It is similar to rms, but defined for signal with finite duration.

   The definition shown in the slice provide two versions: one for continuous time signal, and the second of discrete (or sampled) time signal. Since we will be doing computation on a microprocessor, the discrete time version is actually more useful.

2. Time-shift property of signal is very important. We model this simply by changing the variable t to t-T where T is the delay time.

3. Unit impulse or delta function or Dirac function $\delta(t)$ is one of the most important signal. Combining with idea 2), we can time shift this to any instance as $\delta(t-T)$ and then use multiply operator to take a sample of a signal x(t) at time T. This is called the sampling property of the unit impulse. You will find this very useful to derive what happens to a continue time signal when we

sample it at regular interval.